

# Partial Dynamical SU(3) Symmetry and the Nature of the Lowest K=0 Collective Excitation in Deformed Nuclei

A. Leviatan and I. Sinai

*Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel*

## Abstract

We discuss the implications of partial dynamical SU(3) symmetry (PDS) for the structure of the lowest  $K=0^+$  ( $K = 0_2$ ) collective excitation in deformed nuclei. We consider an interacting boson model Hamiltonian whose ground and  $\gamma$  bands have good SU(3) symmetry while the  $K = 0_2$  band is mixed. It is shown that the double-phonon components in the  $K = 0_2$  wave function arise from SU(3) admixtures which, in turn, can be determined from absolute E2 rates connecting the  $K = 0_2$  and ground bands. An explicit expression is derived for these admixtures in terms of the ratio of  $K = 0_2$  and  $\gamma$  bandhead energies. The SU(3) PDS predictions are compared with existing data and with broken-SU(3) calculations for  $^{168}\text{Er}$ .

21.60Fw, 21.10.Re, 21.60.Ev, 27.70.+q

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The nature of the lowest  $K=0^+$  [ $K=0_2$ ] excitation in deformed nuclei is still subject to controversy. Recently its traditional interpretation as a vibration in the  $\beta$  degree of freedom [1] has been actively discussed and contested [2-5]. The preferential decay of some  $K=0_2$  bands in deformed nuclei to the  $\gamma$  band rather than to the ground ( $g$ ) band have led Casten and von Brentano to suggest that these bands should be understood as phonon excitations built on top of the  $\gamma$  band [2]. Such decay pattern is consistent with calculations [6] in the interacting boson model [7] (IBM) and in the dynamic deformation model [3]. This new interpretation was subsequently questioned and challenged. Burke and Sood have claimed that the observed relative E2 strengths could arise from rather minor double- $\gamma$ -phonon admixtures [4]. Günther *et al.* have argued that the empirical evidence presented in [2] involves higher-spin levels which are sensitive to  $K$  admixtures, and have shown that band mixing calculations can explain the  $K = 0_2 \rightarrow \gamma$  transitions without the assumption of double- $\gamma$ -phonon character [5]. The most relevant information needed to resolve the structure of the  $K = 0_2$  band lies in absolute transition rates. An important step in this debate was therefore the measurement of lifetimes of the lowest  $2_{K=0_2}^+$  level [8] and the measurement via Coulomb excitation of  $B(E2)$  values connecting the  $2_g^+$  and  $2_\gamma^+$  states with the  $0_{K=0_2}^+$  level in  $^{168}\text{Er}$  [9]. This nucleus was recently shown to be a good example of SU(3) partial dynamical symmetry (PDS), for which the ground and  $\gamma$  bands have good SU(3) symmetry, while the lowest excited  $K = 0_2$  band is mixed [10]. The purpose of this work is to study the nature of this band under the assumption of SU(3) PDS and to compare the predictions with the above mentioned  $^{168}\text{Er}$  data and with broken-SU(3) calculations in the IBM framework.

An IBM Hamiltonian with partial SU(3) symmetry has the form [10]

$$H = h_0 P_0^\dagger P_0 + h_2 P_2^\dagger \cdot \tilde{P}_2 \quad . \quad (1)$$

Here  $s^\dagger$  ( $d^\dagger$ ) are monopole (quadrupole) bosons whose total number is  $N$ , the dot implies a scalar product and  $P_0^\dagger = d^\dagger \cdot d^\dagger - 2(s^\dagger)^2$ ,  $P_{2,\mu}^\dagger = 2s^\dagger d_\mu^\dagger + \sqrt{7}(d^\dagger d^\dagger)_\mu^{(2)}$  are boson-pairs,  $\tilde{P}_{2,\mu} = (-1)^\mu P_{2,-\mu}$ . For  $h_0 = h_2$  the above Hamiltonian is an  $SU(3)$  scalar related to the Casimir operator of  $SU(3)$ , while for  $h_0 = -5h_2$  it is an  $SU(3)$  tensor,  $(\lambda, \mu) = (2, 2)$ . Although  $H$  is not an  $SU(3)$  scalar, it has a subset of solvable states with good  $SU(3)$  symmetry. The solvable eigenstates belong to the ground and  $\gamma_{K=2k}^k$  bands and are simply selected members of the Elliott basis [11] with good  $SU(3)$  symmetry,  $(\lambda, \mu) = (2N - 4k, 2k)K = 2k$ . States in other bands are mixed. The partial  $SU(3)$  symmetry of  $H$  is converted into partial dynamical  $SU(3)$  symmetry by adding to it  $O(3)$  rotation terms which lead to an  $L(L + 1)$  splitting but do not affect the wave functions.

The Hamiltonian of Eq. (1) with  $h_0 = 2h_2 = 0.008$  MeV was used in [10] to demonstrate the relevance of  $SU(3)$  PDS to the spectroscopy of  $^{168}\text{Er}$ . The resulting  $SU(3)$  decomposition of the lowest bands is shown in Fig. 1, and compared to the conventional broken- $SU(3)$  calculations of Warner Casten and Davidson (WCD) [12] where an  $O(6)$  term is added to an  $SU(3)$  Hamiltonian, and to the consistent-Q formalism (CQF) [13], where the Hamiltonian involves a non- $SU(3)$  quadrupole operator. In the WCD and CQF calculations all states are mixed with respect to  $SU(3)$ . In the PDS calculation, states belonging to the ground ( $K = 0_1$ ) and  $\gamma$  ( $K = 2_1$ ) bands are pure Elliott states  $\phi_E((2N, 0)K = 0, L)$  and  $\phi_E((2N - 4, 2)K = 2, L)$  respectively, while the  $K = 0_2$  band is mixed and has the structure

$$|L, K = 0_2\rangle = A_1 \tilde{\phi}_E((2N - 4, 2)\tilde{K} = 0, L) + A_2 \tilde{\phi}_E((2N - 8, 4)\tilde{K} = 0, L) \\ + A_3 \phi_E((2N - 6, 0)K = 0, L) . \quad (2)$$

Here  $\tilde{\phi}_E$  denote states orthogonal to the solvable  $\gamma_{K=2k}^k$  Elliott's states. For  $^{168}\text{Er}$  ( $N = 16$ ) the  $K = 0_2$  band contains 9.6%  $(26, 0)$  and 2.9%  $(24, 4)$  admixtures into the dominant  $(28, 2)$  irreducible representation (irrep). Using the geometric analogs of the  $SU(3)$  bands

[14],  $(2N - 4, 2)K = 0 \sim \beta$ ,  $(2N - 8, 4)K = 0 \sim (\sqrt{2}\beta^2 + \gamma^2_{K=0})$ ,  $(2N - 6, 0)K = 0 \sim (\beta^2 - \sqrt{2}\gamma^2_{K=0})$ , the wave function of Eq. (2) can be expressed in terms of the probability amplitudes for single- and double- phonon  $K = 0$  excitations

$$A_\beta = A_1, \quad A_{\gamma^2} = (A_2 - \sqrt{2}A_3)/\sqrt{3}, \quad A_{\beta^2} = (\sqrt{2}A_2 + A_3)/\sqrt{3}. \quad (3)$$

It follows that in the PDS calculation, the  $K = 0_2$  band of  $^{168}\text{Er}$  contains admixtures of 12.4%  $\gamma^2_{K=0}$  and 0.1%  $\beta^2$  into the  $\beta$  mode, *i.e.* 12.5% double-phonon admixtures into the dominant single-phonon component.

General properties of the  $K = 0_2$  band can be studied by examining the general  $\text{SU}(3)$  PDS Hamiltonian of Eq. (1). In Fig. 2 we show the results (filled symbols connected by solid lines) of an exact diagonalization ( $N = 16$ ) as a function of  $h_0/h_2$ . The empirical value of the ratio of  $K = 0_2$  and  $\gamma$  bandhead energies  $E(0_2^+)/[E(2_\gamma^+) - E(2_g^+)] = 0.8 - 1.8$ , in the rare-earth region [2,6] constrains the parameters of H to be in the range

$$0.7 \leq \frac{h_0}{h_2} \leq 2.4. \quad (4)$$

In general the  $K = 0_2$  wave function retains the form as in Eq. (2) and, therefore, a 3-band mixing calculation is sufficient to describe its structure. To gain more insight into this band mixing, we calculate the matrix elements of  $H$  (1) between large- $N$  intrinsic states [15]

$$\begin{aligned} |\beta\rangle &= b_\beta^\dagger |c; N-1\rangle, \quad |\beta^2\rangle = (1/\sqrt{2})(b_\beta^\dagger)^2 |c; N-2\rangle, \quad |\gamma^2_{K=0}\rangle = d_2^\dagger d_{-2}^\dagger |c; N-2\rangle, \\ |c; N\rangle &= (N!)^{-1/2} (b_c^\dagger)^N |0\rangle, \quad b_c^\dagger = (1/\sqrt{3})(s^\dagger + \sqrt{2}d_0^\dagger), \quad b_\beta^\dagger = (1/\sqrt{3})(d_0^\dagger - \sqrt{2}s^\dagger). \end{aligned} \quad (5)$$

To order  $\sqrt{N}$ , the symmetric matrix elements ( $M_{ij}$ ) are

$$\begin{aligned} M_{\beta,\beta} &= M_{\beta^2,\beta^2}/2 = \epsilon_\beta, \quad M_{\gamma^2,\gamma^2} = 2\epsilon_\gamma, \\ M_{\beta,\gamma^2} &= -\sqrt{2}M_{\beta,\beta^2} = -4(h_0 - h_2)\sqrt{N}, \quad M_{\gamma^2,\beta^2} = 0, \\ \epsilon_\beta &= 4(2h_0 + h_2)N, \quad \epsilon_\gamma = 12h_2N. \end{aligned} \quad (6)$$

Diagonalization of the  $3 \times 3$  matrix  $M_{ij}$  provides a good estimate both for the bandhead ratio and for the single- and double-phonon probabilities  $(A_\beta)^2$ ,  $(A_{\gamma^2})^2$ ,  $(A_{\beta^2})^2$ , as shown by the dotted lines in Fig. 2. When the lowest eigenvalue of the matrix  $M_{ij}$  is smaller than both  $2\epsilon_\beta$  and  $2\epsilon_\gamma$ , the eigenvalue equation simplifies, and we can derive the following expressions for the bandhead ratio

$$\begin{aligned} \frac{E(0_2^+)}{E(2_\gamma^+) - E(2_g^+)} &= 1 + y - \frac{1}{4N} y^2 \frac{3+y}{1-y^2} , \\ y = \frac{2}{3} \left[ \left( \frac{h_0}{h_2} \right) - 1 \right] &= \frac{\epsilon_\beta}{\epsilon_\gamma} - 1 , \end{aligned} \quad (7)$$

and for the mixing amplitudes

$$\begin{aligned} A_\beta &= \frac{1}{\sqrt{1+\Delta}} , \quad A_{\gamma^2} = -\frac{1}{\sqrt{2N}} \frac{y}{(1-y)} A_\beta , \quad A_{\beta^2} = \frac{1}{2\sqrt{N}} \frac{y}{(1+y)} A_\beta , \\ \Delta &= \frac{1}{4N} y^2 \left[ \frac{2}{(1-y)^2} + \frac{1}{(1+y)^2} \right] . \end{aligned} \quad (8)$$

These expressions are valid for  $|y| < 1 - 1/\sqrt{2N}$ . The corresponding results of this approximation are shown in Fig. 2 as open symbols connected by dot-dashed lines. For  $^{168}\text{Er}$ , ( $h_0 = 2h_2$ ,  $y = 2/3$ ,  $N = 16$ ), Eq. (7) yields an estimate of 1.62 for the bandhead ratio as compared with the exact value 1.64. From Eq. (8) we obtain a mixing of 11.1%  $\gamma_{K=0}^2$  and 0.2%  $\beta^2$  into the  $\beta$  mode in good agreement with the exact results mentioned above. The quantity  $y$  in Eq. (7) measures, for large  $N$ , the extent to which the  $K = 0_2$  band is above ( $y > 0$ ) or below ( $y < 0$ ) the  $\gamma$  band, and signals the deviation from SU(3) symmetry. In the  $SU(3)$  limit  $y = 0$  ( $h_0 = h_2$ ,  $\epsilon_\beta = \epsilon_\gamma$ ), there is no SU(3) mixing hence no mixing of double-phonon excitations into the  $K = 0_2$  band ( $A_{\gamma^2} = A_{\beta^2} = 0$  in Eq. (8)). In general, the  $SU(3)$  mixing  $(1 - A_\beta^2)$  is  $(1/N)$  suppressed, but the mixing can be large when  $|y| \rightarrow 1$  ( $h_0/h_2 \rightarrow 2.5$ ), corresponding to  $\epsilon_\beta/\epsilon_\gamma \rightarrow 2$ . The SU(3) breaking and double-phonon admixture is more pronounced for  $y > 0$  ( $h_0/h_2 > 1$ ,  $\epsilon_\beta > \epsilon_\gamma$ ). This can be understood from the expression for  $\Delta$  in Eq. (8), which is not symmetric about  $y = 0$ . Near the SU(3) limit

(small  $y$ ),  $(1 - A_\beta^2) \sim \Delta \sim (1/4N)y^2[3 + 2y]$ , which is larger for  $y > 0$ . This implies that the two-phonon admixtures are expected to be larger when the  $K = 0_2$  band is above the  $\gamma$  band. As seen from Fig. 2, for most of the relevant range of  $h_0/h_2$ , Eq. (4), corresponding to bandhead ratio in the range  $0.8 - 1.65$ , the double-phonon admixture is at most  $\sim 15\%$ . Only for higher values of the bandhead ratio can one obtain larger admixtures and even dominance of the  $\gamma_{K=0}^2$  component in the  $K = 0_2$  wave function.

An important clue to the structure of  $K = 0_2$  collective excitations comes from E2 transitions. The relevant operator is

$$T(E2) = \alpha Q^{(2)} + \theta \Pi^{(2)} \quad (9)$$

where  $Q^{(2)}$  is the quadrupole  $SU(3)$  generator and  $\Pi^{(2)} = (d^\dagger s + s^\dagger \tilde{d})$  is a  $(2,2)$  tensor under  $SU(3)$ . Since the wave functions of the solvable states are known, it is possible to obtain analytic expressions for the E2 rates between them [10]. If we recall that only the ground band has the  $SU(3)$  component  $(\lambda, \mu) = (2N, 0)$ , that  $Q^{(2)}$ , as a generator, cannot connect different  $SU(3)$  irreps and that the  $\Pi^{(2)}$  term can connect the  $(2N, 0)$  irrep only with the  $(2N - 4, 2)$  irrep, we obtain the following expressions for B(E2) values of  $\gamma \rightarrow g$  and  $K = 0_2 \rightarrow g$  transitions

$$\begin{aligned} B(E2; \gamma, L \rightarrow g, L') = & \theta^2 \frac{|\langle \phi_E((2N, 0)K = 0, L') | \Pi^{(2)} | \phi_E((2N - 4, 2)K = 2, L) \rangle|^2}{(2L + 1)} \\ B(E2; K = 0_2, L \rightarrow g, L') = & A_\beta^2 \theta^2 \frac{|\langle \phi_E((2N, 0)K = 0, L') | \Pi^{(2)} | \tilde{\phi}_E((2N - 4, 2)\tilde{K} = 0, L) \rangle|^2}{(2L + 1)}. \end{aligned} \quad (10)$$

Here  $\tilde{\phi}_E(\tilde{K} = 0, L)$  is the state orthogonal to the solvable Elliott's state  $\phi_E(K = 2, L)$  in the irrep  $(2N - 4, 2)$ . The Elliott states in Eq. (10) can be expressed in terms of the Vergados basis [16] for which the reduced matrix elements of  $\Pi^{(2)}$  are known [17,18]. The  $E2$  parameter

$\theta$  in Eq. (10) can be determined from the known  $2_{\gamma}^{+} \rightarrow 0_g^{+}$  E2 rates, and for  $^{168}\text{Er}$  is found to be  $\theta^2 = 2.175$  W.u. As seen from Eq. (10), the B(E2) values for  $K = 0_2 \rightarrow g$  transitions are proportional to  $(A_{\beta})^2$ , hence, provide a direct way for extracting the amount of SU(3) breaking and the admixture of double-phonon excitations in the  $K = 0_2$  wave function. In Table 1 we compare the predictions of the PDS and broken-SU(3) calculations with the B(E2) values deduced from a lifetime measurement of the  $2_{K=0_2}^{+}$  level in  $^{168}\text{Er}$  [8] (the indicated range for the B(E2) values correspond to different assumptions on the feeding of the level) and with the B(E2) values connecting the  $2_g^{+}$  and  $2_{\gamma}^{+}$  states with the  $0_{K=0_2}^{+}$  level, measured in Coulomb excitation [9]. It is seen that the PDS and WCD calculations agree well with the lifetime measurement, but the CQF calculation under-predicts the  $K = 0_2 \rightarrow g$  data. This may be due to the fact that the CQF parameters are triggered to spectral properties of the ground and  $\gamma$  bands. On the other hand, all calculations show large deviations from the quoted B(E2) values measured in Coulomb excitation. It should be noted, however, that there are serious discrepancies between the above two measurements. First, Härtelin *et al.* [9], based on their Coulomb excitation measurement and use of generalized Alaga rule, predict a value of  $0.058 \pm 0.007$  (W.u.) for the  $2_{K=0_2}^{+} \rightarrow 0_g^{+}$  transition, which is marginally within the extreme range of the lifetime measurement of Lehmann *et al.* [8]. The latter refers to an extreme and, therefore, highly unlikely feeding scenario. Second, the quoted Lehmann [8] value of 6.2 W.u. (or 3.1 W.u. assuming 50% E2 multipolarity) for the  $2_{K=0_2}^{+} \rightarrow 2_{\gamma}^{+}$  transition, translates via the Alaga rule to a value of 21.7 (or 10.85) W.u. for the  $0_{K=0_2}^{+} \rightarrow 2_{\gamma}^{+}$  transition. The latter is a factor of 7.8 (or 3.9) larger than the value  $2.8 \pm 0.4$  W.u. of Härtelin [9]. An independent measurement of the lifetime of the  $0_{K=0_2}^{+}$  in  $^{168}\text{Er}$  is highly desirable to clarify this issue.

To summarize, we have investigated the nature of the lowest collective  $K = 0$  excitation in deformed nuclei under the assumption of SU(3) partial dynamical symmetry (PDS). We have

presented three types of calculations: an exact diagonalization, a 3-band mixing calculation using intrinsic states, and an analytic approximation to the latter. In this framework, the  $SU(3)$  breaking and double-phonon admixture in the  $K = 0_2$  wave function are intertwined. The mixing is of order  $(1/N)$  but depends critically on the ratio of the  $K = 0_2$  and  $\gamma$  bandhead energies. It can be obtained directly from a knowledge of absolute  $E2$  rates connecting the  $K = 0_2$  band with the ground band. The PDS predictions agree with the lifetime measurement of the  $2^+_{K=0_2}$  level in  $^{168}\text{Er}$  [8] but a noticeable discrepancy remains with respect to the  $B(E2)$  values measured via Coulomb excitation [9]. For the  $K = 0_2$  wave function in  $^{168}\text{Er}$ , we find 12.5% of double-phonon admixtures into the dominant single-phonon component. These findings support the conventional single-phonon interpretation for this band with small but significant double- $\gamma$ -phonon admixture.

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# TABLES

TABLE I. Comparison of theoretical and experimental absolute B(E2) values [W.u.] for transitions from the  $2_{K=0_2}^+$  level [8] and to the  $0_{K=0_2}^+$  level [9] in  $^{168}\text{Er}$ .

Transition	Exp.		Calc.		
	B(E2)	range	PDS	WCD [12]	CQF [13]
Lifetime measurement [8]					
$2_{K=0_2}^+ \rightarrow 0_g^+$	0.4	0.06–0.94	0.65	0.15	0.03
$2_{K=0_2}^+ \rightarrow 2_g^+$	0.5	0.07–1.27	1.02	0.24	0.03
$2_{K=0_2}^+ \rightarrow 4_g^+$	2.2	0.4–5.1	2.27	0.50	0.10
$2_{K=0_2}^+ \rightarrow 2_\gamma^+ \text{ }^a)$	6.2 (3.1)	1–15 (0.5–7.5)	4.08	4.16	4.53
$2_{K=0_2}^+ \rightarrow 3_\gamma^+ \text{ }^a)$	7.2 (3.6)	1–19 (0.5–9.5)	7.52	7.90	12.64
Coulomb excitation [9]					
$2_g^+ \rightarrow 0_{K=0_2}^+$	$0.08 \pm 0.01$		0.79	0.18	0.03
$2_\gamma^+ \rightarrow 0_{K=0_2}^+$	$0.55 \pm 0.08$		3.06	3.20	5.29

<sup>a)</sup> The two numbers in each entry correspond to an assumption of pure E2 and (in parenthesis) 50% E2 multipolarity.

## FIGURES

FIG. 1.  $SU(3)$  decomposition of wave functions of the ground ( $K = 0_1$ ),  $\gamma$  ( $K = 2_1$ ), and  $K = 0_2$  bands of  $^{168}\text{Er}$  ( $N = 16$ ) in the  $SU(3)$  PDS calculation (present work), and broken- $SU(3)$  calculations WCD [12] and CQF [13].

FIG. 2. Properties of the  $K = 0_2$  band as a function of  $h_0/h_2$ , parameters of the  $SU(3)$  PDS Hamiltonian, Eq. (1),  $N=16$ . (a) Ratio of  $K = 0_2$  and  $\gamma$  bandhead energies obtained from an exact diagonalization (filled circles), 3-band mixing calculation based on Eq. (6) (dotted line) and an approximation based on Eqs. (7)–(8) (open circles connected by a dot-dashed line). (b) Probability amplitudes squared,  $(A_\beta)^2$  (circles),  $(A_{\gamma^2})^2$  (squares),  $(A_{\beta^2})^2$  (triangles down) for the  $K = 0_2$  wave function. Notation for the different curves as in part (a) with corresponding symbols.



